Reversing Education Majors' Arithmetic Misconceptions With Short-Term Instruction Using Manipulatives

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ABSTRACT The authors examined the impact of manipulative-based instruction on 2 independent cohorts of preservice elementary teachers. In Study 1, 50 participants engaged in problem solving with operations on whole numbers and fractions using concrete and representational manipulatives over 5 classes. Pre- to posttest performance on a mathematics survey and division-of-fractions task showed significant improvement in arithmetic knowledge, accompanied by a significant decrease in arithmetic misconceptions. Study 2 replicated Study 1 findings with an independent sample of 39 volunteers. These results indicate that manipulatives can effectively and efficiently reverse long-standing arithmetic misconceptions and increase arithmetic knowledge before education majors enter classrooms as full-time teachers.

Keywords: mathematics misconceptions, mathematics misunderstandings, teacher training

Lack of mathematics sophistication in preservice teachers has been addressed by the teaching principle set forth in the Principles and Standards for School Mathematics (National Council of Teachers of Mathematics [NCTM], 2000). According to the principle,

> Teachers must know and understand deeply the mathematics they are teaching. . . . They need to . . . be skillful in choosing from and using a variety of pedagogical and assessment strategies. . . . They need to know the ideas with which students often have difficulty and ways to help bridge common misunderstandings. (p. 17)

A recurrent barrier to deep understanding and skillful pedagogy is that most teachers teach the same way they were taught in elementary and high school rather than as they learned in undergraduate teacher-education programs (Bauersfeld, 1998; Désautels, 2000). Most preservice teachers display little change in underlying dispositions or beliefs established long before they entered college (Zeichner & Gore, 1990). Such a stagnant condition is exacerbated when teachers hold arithmetic misconceptions (i.e., false knowledge believed to be true knowledge) that could ultimately mislead one or two generations of new students. To advance teachers' mathematics understanding and improve instructional pedagogy, mathematics educators must find effective ways to reverse well-documented teacher misconceptions about arithmetic.

We discuss arithmetic misconceptions in preservice elementary teachers and past attempts to remediate them. We also discuss use of manipulatives for teaching arithmetic because it is relevant for remediating misconceptions of preservice teachers. Finally, we report on two studies that test and replicate the effectiveness of an in-class intervention designed to reverse common arithmetic misconceptions and improve arithmetic understanding.

Misconceptions About Arithmetic

The most extensive recent work on teachers' arithmetic misconceptions has been reported by Graeber, Tirosh, and Glover (1989) and Tirosh and Graeber (1989, 1990a, 1990b). The significance of that body of work is twofold: First, it clearly identified the prevalence of misconceptions; second, it pioneered an effective procedure for addressing the problem. Yet, in over a decade since the original reports, relatively little research on teacher misconceptions has been published.

In the original study on teachers' mathematics misconceptions, Tirosh and Graeber (1989) revealed that many preservice teachers have difficulty selecting the appropriate operation for multiplication and division word problems, even though they can correctly compute with standard algorithms for numbers less than 1. One reason for the difficulty is the false belief that a quotient must be less than a dividend. Teachers have reported other patterns of misconception thinking:

1. Misconceptions occur with all four arithmetic operations (e.g., addition and multiplication make numbers

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larger; subtraction and division make numbers smaller; Graeber et al., 1989; Tirosh & Graeber, 1989).

2. Mastery of computational algorithms makes misconceptions more resilient (Tirosh & Graeber, 1990a).

3. Misconceptions are resilient to correction even when disconfirming computations and incompatible information are given (Fischbein, Deri, Nello, & Marino, 1985; Tirosh & Graeber, 1989).

Remediating Misconceptions

Researchers have used cognitive conflict (reflective abstraction) to try to reverse teachers' arithmetic misconceptions. For example, Tirosh and Graeber (1990a) followed Swan's (1983) lead in using conflict teaching to involve preservice teachers in discussions about their misconceptions. The researchers believed that discussions would result in cognitive conflict because of inconsistencies between stated misconceptions and results of computations. Although positive results occurred, cognitive conflict could not be induced in a few of the preservice teachers.

Tirosh and Graeber's (1989, 1990a, 1990b) misconception reports described landmark studies, but there were conceptual limitations. First, conflict-teaching interviews served as independent and dependent variables, thereby confounding the treatment with their resulting effect.

Second, inducing cognitive conflict can often be a risky, inefficient enterprise because individuals seldom recognize their own misconceptions. For example, of 21 participants, Tirosh and Graeber (1990a) reported that only 6 students spontaneously recognized a contradiction and 7 students recognized a contradiction only after multiple, simple prompts. Of the remaining 8 students, 1 student never realized an inherent contradiction between her computed solution and her division misconception, and the 7 other students needed multiple prompts and problem situations to achieve cognitive conflict. In spite of these variations in cognitive conflict, Tirosch and Graeber did not report any analyses about degree of cognitive conflict achieved or subsequent performance in or knowledge of mathematics. The interviewing sophistication needed to achieve this level of cognitive conflict would be difficult to replicate in teacher-education programs even if conflict teaching became a universal remedy for preservice teachers' misconceptions.

Third, Tirosh and Graeber's (1990a) one-on-one interviews lasted 35 to 50 min and occurred only in the context of division. Extrapolated to include addition, subtraction, and multiplication, such an intervention could rapidly become prohibitively labor intensive. Conflict teaching is probably not an intervention that has widespread efficacy for teacher-education faculty, particularly because teachers should never have learned misconceptions initially.

Tirosh and Graeber (1990a) reported that teachers' mastery of computational algorithms made their misconceptions more resilient; Ball's (1990a, 1990b) research bears on this point. She reported that preservice elementary and secondary teachers could correctly compute a division-of-fractions problem but could not accurately portray the meaning of the solution. Moreover, even teachers' correctly drawn pictures were of the problem statement rather than the solution. Only 1 of Ball's 19 students focused on explaining division; the remaining 18 focused on only the number that was being divided. Similarly, Piel and Green (1994) reported that sixth-grade students and preservice elementary teachers were equally competent at computing division of fractions (e.g., 1/2 ÷ 1/3 = 1 1/2). Conceptually, neither group understood that the solution 1 1/2 referred to 1 1/2 one thirds. Most teachers and students thought the solution referred to 1 1/2 wholes. Piel and Green argued that sixth graders who memorize rules such as invert and multiply for dividing fractions may grow up like the teachers Ball (1990a, 1990b) studied who computed successfully but without understanding.

Manipulatives

Although results differ depending on what and how manipulatives are used in learning situations, learning with manipulatives is correlated positively with later development of mental mathematics (Gravemeijer, 1990), achievement, and understanding (Sowell, 1989). For example, base-10 blocks (a popular mathematics manipulative that contains small units for ones, thin rods for tens, ten-by-ten flats for hundreds, and a large ten-by-ten-by-ten block for thousands place values) improves students' conceptual understanding of arithmetic operations (Carpenter et al., 1999; Fuson & Briars, 1990). In geometry, using a compass to learn about circles produces better understanding of center and radius concepts than does traditional circle tracings and templates (Chassapis, 1999).

If manipulatives produce better conceptual understandings than do traditional teaching techniques, why are they not typically used for teaching mathematics to preservice teachers? Successful completion of college mathematics courses does not ensure that taking higher level mathematics influences previously constructed misconceptions. One reason may be that teachers tend to use manipulatives as enjoyable diversions but do not believe they are essential to teaching and understanding. Given their effectiveness with young students, is it not possible that relearning with manipulatives could undo or reverse misconceptions previously learned by preservice teachers? We designed the two studies reported here to answer this question.

Rationale

Piaget's constructivism is best captured in a fundamental tenet: To understand is to invent. Accordingly, conceptual knowledge originates in the inventive activities of the learner through actions on objects rather than from sensory impression or social transmission derived from teachers and parents (Piaget, 1970, 1974). Following this reasoning, arithmetic misconceptions can be understood as inventions and meanings that reflect incomplete or partially adapted knowledge.

The widespread and well-documented arithmetic misconceptions among preservice education majors suggest common but incomplete school experiences spent learning arithmetic with, about, and through symbols. We suggest that misconceptions may originate and persist because they depend purely on symbols and do not affect computational accuracy. Simple misconceptions learned in elementary school (e.g., addition and multiplication make larger; subtraction and division make smaller) may produce conceptual misunderstandings that seldom get corrected in more complex mathematics classes because even when symbolic answers are computed accurately, the meaning of a solution is seldom questioned.

Support for this line of reasoning derives from Tirosh and Graeber (1990a), who suggested that a standard algorithm may be the source and continuing support for an underlying misconception. In this sense, a misconception may originate in a conceptual limitation of abstract number symbols. That is, when numerals instead of objects undergo arithmetic manipulations, as is typical in elementary school classrooms, one can easily see how inappropriate conclusions can be drawn as a function of the numerals used rather than the values they represent.

Table 1 compares sample compositions and solutions for symbolic and concrete manipulatives for the four arithmetic operations. If students are taught by teachers using only symbolic notation (numerals), they may perceive that addition and multiplication do make numbers bigger because the numeral on the right side of the equal sign is larger than either factor or number on the left. Similarly, subtraction and division operations produce numerals on the right side of the equal sign that are smaller than either of the factors on the left.

Conversely, cells in the counters (round counting disks) column represent number of objects rather than number of numerals. For addition problems, counters on the left equal the counters on the right; the equation is balanced. For every arithmetic operation, concrete objects make clear that one cannot obtain more or fewer objects than one started with, unlike that which computational algorithms do with symbolic numerals. For example, 6 ÷ 3 = 2 in symbolic notation represents an arrangement of 6 grouped into sets of 3 two times, which is depicted by the counters. With counters, the solution 2 is visually compelling because it shows two sets of three. Using counters, one might conclude that division, like all arithmetic operations, is a way of arranging objects (e.g., disaggregating by groups of equal size). This line of reasoning suggests that concrete manipulatives may provide an effective way to remediate misconceptions in preservice teachers.

The pedagogical strategy described in the previous paragraph does not depend on verifying the presence of cognitive conflict as a psychological state of disequilibrium in an individual's mind nor does it depend on reflective abstraction (Piaget, 1974); rather, our primary focus is on performing actions on objects with a hierarchy of manipulatives. In this sense, our treatment might be called guided constructivism. Students are guided in building an appropriate concrete model and then shown how to use the model to perform actions embedded in problem situations. Because our focus is action on objects, the first two authors teach action language that describes an arithmetic arrangement to be performed for any real-world problem. With guided constructivism, students learn to use manipulatives as tools of action that can be later incorporated into mental actions on objects.

Our notion of guided constructivism contrasts with what Simon (2000) referred to as radical constructivism, the subjective construction of personal and social meanings. Guided constructivism is conceptually similar to Freudenthal's (1991) guided reinvention or guided reconstruction. Our intent was to guide students through a hierarchy of manipulatives that represent increasingly abstract objects—from concrete to representational to transitional to symbolic—by using manipulatives as tools for examining, constructing, seeing, and testing the performance of arithmetic operations and studying their interrelationships. For example, addition is the same action of joining whether the joining is of Cuisenaire rods, steps on a number line, tally marks in a place-value chart, symbolic numerals, or even variables—joining is always the same action regardless of the objects that are joined.

One can contrast guided constructivism with Bruner's (1960) concept of discovery learning and its attendant spiral curriculum. Underlying Bruner's approach is the belief that "learning depends upon knowledge of the results of one's tests, and instruction should have an edge over 'spontaneous' learning in providing more of such

TABLE 1. Comparison of Visual Information Obtained Through Symbolic Notation and Hands-On Manipulatives

Operation	Symbolic notation	Hands-on manipulative (counters)
Addition	2 + 3 = 5	oo + ooo = ooooo
Multiplication	2 × 3 = 6	ooo ooo = ooooo o
Subtraction	5 – 3 = 2	oo (ooo) → = oo
Division	6 √ 3 = 2	ooooo o √ ooo = ooo ooo

knowledge" (p. 44). Therein, students are charged with learning content and skills of a discipline, and a good teacher is able to teach any subject "to anybody at any age in some form" (Bruner, p. 12). In contrast, guided constructivism begins with problems to solve, invokes the use of manipulatives as tools for problem solving, and relies on the invention of personal meanings as the basis for understanding and sense making.

For purposes of the studies reported here, we hypothesized that classroom instruction using concrete and representational manipulatives with preservice teachers would produce (a) decreased arithmetic misconceptions (invalid knowledge) and (b) a corresponding increase in correct knowledge.

Method

Participants

Participants in Study 1 were 53 undergraduate volunteers enrolled in three sections of a child-development course taught by the first two authors. The university's institutional review board approved data-collection procedures in which the teaching professor left the classroom and in his absence a nonteaching author solicited volunteers and collected data from active participants. Participation in the study was kept confidential from the teaching professor until after the end of the semester. The child-development course is a program requirement for elementary education majors at a large state university in the Southeast accredited by the National Council for Accreditation of Teacher Education. Students generally take the course in the first semester of their junior year. Most of the students were women (88.7%) with an average age of 26.12 years (SD = 7.88). The average SAT mathematics score was 487.27 (SD = 67.05).

Treatment

To illustrate how constructivism can be applied to elementary classrooms, course instructors incorporated abbreviated lessons with concrete and representational manipulatives adapted from their mathematics education class. The lessons covered 4 classes of a 30-class semester. All instruction was oriented toward classes rather than individuals. Table 2 provides an outline of the content of these classes, which consisted of using manipulatives to solve a problem posed by the instructor. Students compared solutions, and the instructor demonstrated the solutions with overhead manipulatives or whiteboard illustrations.

The instructors had co-taught an elementary mathematics methods course for the past 10 summers and were well-acquainted with each others' examples, demonstrations, and pacing. They agreed on a sequence of lessons (activities and demonstrations) and the timing for various instructional components. Total instruction time was four 1 1/3 hr class sessions (5 1/3 hr; see Table 2).

Table 2 shows that students sometimes actively engaged in activities and sometimes simply observed the instructor solving problems with manipulatives. All participants solved whole-number addition and multiplication problems with Cuisenaire rods and base-10 blocks, then observed the instructor's use of a place-value chart, expanded notation, and partial sums and products algorithms. For subtraction and division of whole numbers and the four arithmetic operations with fractions, students used only Cuisenaire rods and observed no higher level manipulations. Instructors did not directly answer students' questions about correct solutions or pedagogical questions. Instead, instructors redirected such questions to the class, to arrangements of manipulatives, or to whiteboard examples of problem solving.

Without directly helping students overcome their misconceptions, instructors managed class discussions with prompts such as:

- Do we end up with more, less, or the same amount that we started with?
- What amount remains on the left side of the equal sign? What amount on the right side?
- Can you ever get more or less than you started with?
- Is the equation balanced? What is a balanced equation?
- Is it important to balance an equation? Why?
- What does the answer represent? (particularly important with division of fractions)
- What does addition (multiplication, subtraction, division) actually do? (Instructors called students' attention to ways of arranging objects: How does the problem tell us the objects should be arranged? When we multiply, do we arrange objects the same way as when we add/subtract them?)

An important component of our guided constructivism was introducing students to and practicing action language for arithmetic operations. Action language provides a ready-made instrument for assimilating the action of a problem, a way of understanding the type of arrangement that is described in a problem. Rather than use traditional arithmetic language that carries few semantic cues for arranging objects, instructors consistently used the following terms to emphasize how an operation describes actions on objects, whether manipulatives or symbols.

Addition	*joined with*	(+)	3 *joined with* 4 is what?
Multiplication	*sets of*	(×)	2 *sets of* 3 is what?
Subtraction	*take away*	(–)	9 *take away* 4 is what?
Division	*grouped into*	(÷)	8 *grouped into* sets of 2 how many times?
		(÷)	8 *grouped into* 2 sets, with how many in each set?

Instructors phrased and modeled all problem situations and open-number sentences using this action language and explicitly made connections for these actions between operations on whole numbers and operations on fractions.

TABLE 2. Instructional Plan for Manipulatives and Content Covered During Treatment

Hierarchy level	Arithmetic content taught: Addition and multiplication of whole numbers	Subtraction and division of whole numbers	Fractions
Symbolic	NA	NA	NA
Transitional	Expanded notation[a]		
	Partial sums, products[a]	NA	NA
Representational	Base 10 blocks,[b] place value chart[a]	NA	NA
Concrete	Cuisenaire rods[b]	Cuisenaire rods	Cuisenaire rods
Instructional time	2 hr	1.7 hr	1.6 hr

Note. NA = not applicable.
[a]Indicates students observed modeling by instructor but may or may not have taken notes.
[b]Indicates students actively participated and solved problems with their kits.

Instrument

The mathematics survey consisted of 20 items based on the research literature and included correct responses and misconception responses about operations on whole numbers and fractions, the meaning of the equal sign, algorithmic procedures, positional meanings in open-number sentences, and conceptual understanding of computing algorithms. We determined evidence of the instrument's validity by using expert judgments of three mathematics education faculty about the relationship between individual items and the constructs of misconceptions and knowledge, as well as the representativeness of the chosen set of items (see American Educational Research Association, American Psychological Association, & National Council on Measurement in Education, 1999).

We calculated our scales from the mathematics survey items: (a) Misconception, (b) Tirosh–Graeber (TG)-Misconception, (c) Knowledge, and (d) TG-Knowledge. We used 15 of the items to calculate the Misconception scale. The misconception score was the percentage of misconception options participants selected across the 15 items (Cronbach's alpha of .87) and resulted in scores ranging from 0% to 100%. We designed 4 of the 15 misconception items to specifically assess misconceptions reported by Tirosh and Graeber (1990a, 1990b) for whole-number operations; TG-Misconception scores were the percentage of misconception options selected on the 4 items (Cronbach's alpha of .89). We calculated the Knowledge scale from the percentage of correct responses according to 16 of the mathematics survey items (Cronbach's alpha of .88). The TG-Knowledge score (Cronbach's alpha of .89) was the percentage of correct responses based on the same 4 items as the TG-Misconception scale, but the TG-Knowledge score reflected correct answer selections rather than misconception answer selections. Because we based the four scales on some common items, there is a dependency among the scale scores. To control for an inflated Type 1 error rate caused by the anticipated relationship between variables, we used a more stringent level of significance (nominal α of .01).

At the end of the mathematics survey, students completed an additional sheet showing a division-of-fractions exercise: 1 1/2 ÷ 3/4 = ? Directions on the sheet prompted students to (a) show the computation of a correct solution and (b) draw a representation of their solution to show the meaning. To assist with an appropriate, easy-to-interpret representation, instructors told students, "Imagine that the problem above is about pizzas. Draw a picture showing how your solution would look in terms of pizzas (label any elements for clarity)."

We scaled computations as follows: 1 (no computation), 2 (incorrect computation), and 3 (correct computation). Pictorial representations were scored on a 4-point scale: 1 (no representation), 2 (inappropriate representation of wrong answer), 3 (inappropriate representation of two objects), and 4 (appropriate representation of two sets of three fourths). Essential to an appropriate representation was an illustration that clearly conveyed the idea of two sets of three fourths pizza (not two whole pizzas). Figure 1 shows examples of inappropriate and appropriate illustrations. The mathematics survey and division-of-fractions problem took students between 8 and 20 min to complete.

Study 1 Results

We included 50 of 53 participants who completed the pretest and the posttest in our data analyses. (Three of the original participants were eliminated because of a missing pretest or posttest.) We conducted four dependent t tests for related samples to examine differences between pretest and posttest in the Misconception, TG-Misconception, Knowledge, and TG-Knowledge scales. Table 3 shows the means, standard deviations, and t tests for these variables.

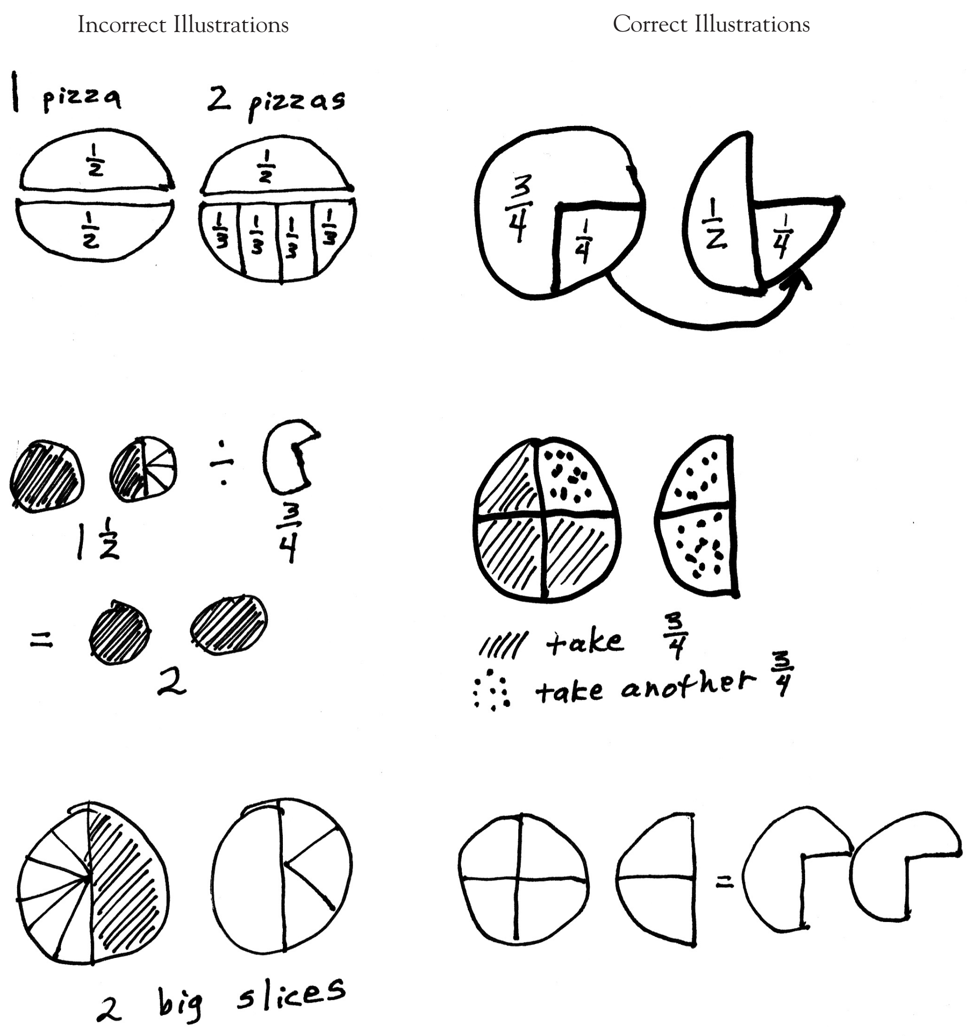

FIGURE 1. Examples of students' incorrect and correct illustrations.

TABLE 3. Study 1 With Four Arithmetic Scales: Mean Pretest and Posttest Percentages, Standard Deviations, and Dependent *t* Test Results

Scale	Pretest (N = 50) M	Pretest (N = 50) *SD*	Posttest (N = 50) M	Posttest (N = 50) *SD*	*t*	*ES*
Misconception	64.27	8.21	14.18	11.77	–26.84*	7.31
TG-Misconception	89.50	20.26	20.00	29.88	–13.33*	4.04
Knowledge	21.97	7.73	72.80	13.46	23.90*	8.76
TG-Knowledge	10.00	20.20	72.50	35.45	10.75*	4.10

Note. Effect size (*ES*) is based on a pooled standard deviation and correction for the relationship between pretest and posttest (Lipsey, 1990).
*$p < .001$.

There was a statistically significant decrease between the pretest and posttest for both misconception scales. For the full Misconception scale, participants had a pretest mean of 64.27% selection of misconception options across the 15 items; but only 14.18% selection of misconception options on the posttest, $t(49) = –26.84$, $p < .01$. On the TG-Misconception scale, participants' mean decreased from 89.50% on the pretest to 20.0% on the posttest,

$t(49) = -13.33$, $p < .01$. The magnitude of change from pretest to posttest for both misconception scales was large. Using Lipsey's (1990) method, we found effect sizes of 7.31 for the Misconception scale and 4.04 for the TG-Misconception scale; findings were similar for both knowledge scales. That is, there was a statistically significant increase in the mean percentage correct response in the Knowledge scale, with a pretest mean correct score of 21.97%, increasing to a posttest mean of 72.80% on the posttest, $t(49) = 23.90$, $p < .01$. The TG-Knowledge percentage correct increased from 10% on the pretest to 72.5% on the posttest, $t(49) = 10.75$, $p < .01$.

One of the most important results of this study was the performance illustration in the division-of-fractions problem, 1 1/2 ÷ 3/4 = ? in which participants exhibited different but predictable patterns. When computing division of fractions, 88.7% of our preservice elementary teachers showed a correct calculation and solution on the pretest. However, the modest improvement to 94.1% correct solutions on the posttest was not significant as computed by the Wilcoxin signed ranks test ($z = .75$, $p = .45$). These results suggest that most participants could accurately compute division of fractions before our intervention. In contrast, 7 participants (14.0%) correctly illustrated two sets of three fourths pizzas on the pretest, but this number improved to 37 participants (74.0%) on the posttest, over a five-fold increase. This pre- to posttest improvement in illustration accuracy was statistically significant ($z = 5.40$, $p < .01$).

Study 2 Results

Thirty-nine participants completed the pretest and posttest in Study 2. Table 4 shows the means, standard deviations, and dependent *t* tests results for the four mathematics scales. The trend in the data was similar to Study 1. A statistically significant decrease occurred between the pretest and posttest for both misconception scales. For the full Misconception scale, participants had a pretest mean of 45.92% selection of misconception options across the 15 items but only 18.17% on the posttest, $t(38) = -13.85$, $p < .01$. On the TG-Misconception scale, participants' mean decreased from 37.18% on the pretest to 12.82% on the posttest, $t(38) = -6.53$, $p < .01$. The magnitude of change from pretest to posttest for both misconception scales was not as large as in Study 1, with effect sizes of 3.22 for Misconception and 1.56 for TG-Misconception. We found similar results for both knowledge scales. Specifically, a statistically significant increase occurred in the mean percentage correct response in the Knowledge scale, with a pretest mean correct score of 17.18% rising to 54.28% on the posttest, $t(38) = 14.79$, $p < .01$. The TG-Knowledge percentage correct increased from a pretest mean of 8.97% to a posttest mean of 55.77%, $t(38) = 6.74$, $p < .01$. Corresponding effect sizes for these results were 4.38 for Knowledge and 2.35 for TG-Knowledge scales.

The results of Study 2 participants on the two performance-based items were similar to those of participants in Study 1. For computing division of fractions, 61.5% of the preservice elementary teachers showed a correct calculation and solution on the pretest. Computational accuracy increased to 71.8% correct on the posttest. However, this increase was not statistically significant ($z = 1.15$, $p = .25$). In contrast, only 6 participants (15.4%) correctly illustrated two sets of three fourths pizzas on the pretest; on the posttest, 22 participants (56.4%), nearly a threefold increase, correctly illustrated the pizzas. This pre- to posttest improvement in illustration accuracy was statistically significant ($z = 3.58$, $p < .01$).

Discussion

The findings of the two studies reported here were statistically significant and educationally poignant. They suggest that a relatively modest investment of instructional time including appropriate manipulative-based problem solving can produce powerful educational benefits for preservice elementary teachers. The primary focus of Study 1 was assessing the hypothesis that short-term arithmetic instruction using hands-on manipulatives and observations of instructor activities with manipulatives can effectively

TABLE 4. Study 2 With Four Arithmetic Scales: Mean Pretest and Posttest Percentages, Standard Deviations, and Dependent *t*-test Results

	Pretest (*N* = 50)		Posttest (*N* = 50)			
Scale	*M*	*SD*	*M*	*SD*	*t*	*ES*
Misconception	45.92	12.86	18.17	11.53	–13.85*	3.22
TG-Misconception	37.18	21.36	12.82	22.85	–6.53*	1.56
Knowledge	17.18	11.54	54.28	15.23	14.79*	4.38
TG-Knowledge	8.97	21.06	55.77	41.94	6.74*	2.35

Note. Effect size (*ES*) is based on a pooled standard deviation and correction for the relation between pretest and posttest (Lipsey, 1990).
*$p < .001$.

reverse arithmetic misconceptions. However, such a finding could have been achieved without corresponding improvement in arithmetic understanding. Consequently, we tested the hypothesis that problem solving with manipulatives can produce more accurate arithmetic knowledge to replace the misconceptions. Both hypotheses were accepted in Study 1 and replicated with an independent sample in Study 2.

The two studies reported are important for four reasons. First, they indicate that the arithmetic misconceptions reported by Tirosh and Graeber (1989, 1990a, 1990b) are still prevalent. Our mathematics survey pretest confirms the continuing frequency of arithmetic false beliefs among preservice elementary teachers.

Second, Study 1 and Study 2 show that manipulatives can effectively reverse most arithmetic misconceptions of elementary education majors before they enter classrooms as full-time teachers. This finding is important because teachers tend to teach the way they were taught (Bauersfeld, 1998; Désautels, 2000). Our results suggest an effective approach to breaking the long-term resistance to change reported by Zeichner and Gore (1990). The biggest pedagogical problem is to help teacher candidates to recognize that they do not know what they do not know. Whereas Tirosh and Graeber (1990a) used one-on-one cognitive conflict training to induce recognition of misconceptions, whole-group instruction with manipulatives can accomplish the same recognition far more efficiently. We believe that participants' active use of manipulatives to solve problems was critical to their reconstructions (no direct, explicit instruction about misconceptions was given). Had we demonstrated only how to use these manipulatives to solve problems without students' direct problem solving, we believe students would not have produced knowledge gains or decreased misconceptions.

Third, our two studies demonstrate that the same activities used to reverse misconceptions can also improve the accuracy and depth of arithmetic knowledge. Given the construction of our mathematics survey, elementary education majors could have reduced misconceptions and replaced them with distracters that sounded appropriate but did not convey accurate knowledge. That did not happen. Our students consistently improved their arithmetic knowledge in both studies. To amplify this point, the computational accuracy for division of fractions showed no pre- to posttest change. This finding is exactly what one would expect from computationally proficient undergraduates. A typical student learns this algorithm in fifth grade and thereafter experiences many years of schooling in which repeated computational practice is an embedded feature of pedagogy. Consequently, if a student can calculate proficiently and consistently derive correct solutions, there is often little opportunity in later algebra, geometry, or calculus coursework to realize or discover any faulty arithmetic misconceptions.

In contrast to fraction computations, only 7 participants in the Study 1 pretest and 6 participants in the Study 2 pretest (15% of all participants) could appropriately illustrate their division-of-fractions solution as two sets of three fourths; most students illustrated two whole pizzas or two pieces of pizza (cf. Ball, 1990a, 1990b; Piel & Green, 1994). During the treatment, students spent only 1 1/2 class periods using Cuisenaire rods to explore operations on fractions. Yet, in this brief treatment, 37 Study 1 participants and 22 Study 2 participants (66% of all participants) provided an appropriate posttest illustration, more than a four-fold improvement in participants' understanding. We suspect that the hands-on instruction was effective in part because, unlike performing calculations on numeric symbols, using Cuisenaire rods enabled students to see the pieces. Students were asked to illustrate their solution to 1 1/2 ÷ 3/4 = ? (see Figure 1). In pretests for both studies, the most frequent illustration showed either two whole pizzas or two pieces of pizza (instead of two sets of three fourths pizza, the correct answer). However, posttest illustrations in both studies showed that although Solution 2 did not change, its meaning had changed for many students. By the posttest, 66% of all participants had exhibited their understanding as two sets of three fourths pizza. With numeric symbols, Solution 2 may have been misleading because it seemed to have no objects to refer to; when students used the standard algorithm, the original three fourths had been transformed symbolically to its inverse, four thirds.

Fourth, the design of the studies reported here, a replication across an independent sample, is unusual in education research literature. As the U.S. Department of Education (2001, 2003) promotes scientifically based evidence and prefers experimental and quasi-experimental studies, replications such as the one reported here may gain support.

If knowledge can be constructed, then it can be reconstructed. The key issue is how mathematics educators can effectively teach education undergraduates to reconstruct more meaningful and accurate understandings of arithmetic concepts. Our two studies show that a cost- and time-effective strategy can be implemented through the use of manipulatives that intrudes modestly into the normal preservice teacher curriculum. The outcome can produce benefits that are immediate, theoretically sound, and pedagogically significant.

Two limitations of our studies were the use of convenience samples and the absence of a control or comparison group. An important and relevant question not examined here concerns the durability and stability of changes in arithmetic knowledge and misconceptions that resulted from short-term interventions. Do elementary education majors remember their learning 1 to 2 years after their experience? A longitudinal study could answer this question.

The problem with misconceptions is that people do not know they have them; therefore, misconceptions may be perpetuated in the classroom. Mathematics educators have a responsibility to engender in teachers a deep understanding of arithmetic—its patterns, functions, and meanings (NCTM, 2000). In our two studies, we suggest there are direct, pragmatic, replicable, and effective means of providing mathematical experiences for preservice education teachers that reduce misconceptions and increase knowledge.

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